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DETERMINATION OF BOUNDS FOR THE SOLUTIONS TO THOSE BINARY DIOPHANTINE EQUATIONS THAT SATISFY THE HYPOTHESES OF RUNGE'S THEOREM*

BY

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ABSTRACT. In 1887 Runge [13] proved that a binary Diophantine equation $F(x, y) = 0$, with F irreducible, in a class including those in which the leading form of F is not a constant multiple of a power of an irreducible polynomial, has only a finite number of solutions. It follows from Runge's method of proof that there exists a computable upper bound for the absolute value of each of the integer solutions x and y. Runge did not give such a computation. Here we first deduce Runge's Theorem from a more general theorem on Puiseux series that may be of interest in its own right. Second, we extend the Puiseux series theorem and deduce from the generalized version a generalized form of Runge's Theorem in which the solutions x and y of the polynomial equation $F(x, y) = 0$ are integers, satisfying certain conditions, of an arbitrary algebraic number field. Third, we compute bounds for the solutions $(x, y) \in \mathbf{Z}^2$ in terms of the height of F and the degrees in x and y of F.

1. Introduction. Runge [13] established that certain binary Diophantine equations have only finitely many solutions. We formulate his hypotheses by introducing the following definitions for a polynomial

$$F(x, y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x^i y^j$$

of degree d_1 and d_2 in x and y respectively.

1.1. DEFINITIONS. Let $\lambda > 0$. Then $F_\lambda(x, y)$, the *λ-leading part of F*, is the sum of all terms $a_{ij}x^i y^j$ of F for which $i + \lambda j$ is maximal. The *leading part of F*, denoted by $\tilde{F}(x, y)$, is the sum of all monomials of F which appear in any F_λ as λ varies.

The sum of the terms of maximal degree, which in our notation is F_1, is the *leading form* or *leading homogeneous part* of F.

If F factors into polynomial factors A, B, then $F_\lambda = A_\lambda B_\lambda$, although in general $\tilde{F} \neq \tilde{A}\tilde{B}$.

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* It is with great regret I announce that my teacher, Ph.D. thesis advisor, and friend, Ernst Gabor Straus, died on July 12, 1983 as a result of a heart attack. His departure will create a void that will be felt throughout the mathematical community. It will be particularly felt by those of us who have had the unique opportunity of studying and publishing papers with Professor Straus.

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As usual, **Z** denotes the integers, **Q** the rationals and **C** the complex numbers. The Cartesian product of a set S with itself is S^2 and $S[x]$, $S[x, y]$ denote the sets of polynomials, with coefficients in S, in x and x, y, respectively.

We can now formulate Runge's hypotheses.

1.2. DEFINITION. Let $F(x, y) \in \mathbf{Z}[x, y]$ be irreducible in $\mathbf{Z}[x, y]$. Then F satisfies *Runge's Condition* unless there exists a λ so that $\tilde{F} = F_\lambda$ is a constant multiple of a power of an irreducible polynomial in $\mathbf{Z}[x, y]$.

RUNGE'S THEOREM. *If F satisfies Runge's Condition, then the Diophantine equation $F(x, y) = 0$ has only finitely many solutions $(x, y) \in \mathbf{Z}^2$.*

An independent proof of Runge's Theorem was given by Skolem [**17**].

In §2 we reprove Runge's Theorem in a manner resembling his original proof. However, we prove that a general class of Puiseux series with sufficiently many initial algebraic coefficients represent algebraic functions whose defining equation violates Runge's Condition whenever its curve passes through infinitely many lattice points $(x, y) \in \mathbf{Z}^2$. We then deduce Runge's Theorem by applying this result to the Puiseux series expansion about infinity of the algebraic function y defined by $F(x, y) = 0$.

We next genralize the Puiseux series theorem to curves which pass through infinitely many algebraically integral lattice points, subject to suitable restrictions (see also Hilliker and Straus [**6**]). We then obtain a corresponding generalization of Runge's Theorem for binary Diophantine equations over an algebraic number field.

In §3 we compute bounds for the solutions $(x, y) \in \mathbf{Z}^2$ for some special cases of those Diophantine equations $F(x, y) = 0$ that satisfy Runge's Condition. We develop a sufficient condition for all the coefficients in the Puiseux series expansion

$$y = a_{-m}x^{m/e} + a_{-m+1}x^{(m-1)/e} + a_{-m+2}x^{(m-2)/e} + \cdots$$

$(m, e \in \mathbf{Z}, e > 0, |x| > R)$ to be in the field of the leading coefficient a_{-m}, and treat this case (Theorem 3.31).

In §4 we treat the general case, which involves more computational complexity, where the Puiseux coefficients need not be in the field of the leading coefficient. We compute a bound (Theorem 4.9) valid for all Diophantine equations that satisfy Runge's Condition.

Hilliker [**3, 4**] has developed techniques for solving certain Diophantine equations. To illustrate those techniques the quartic case of Runge's Theorem has been developed (Hilliker [**4**]). The methods of the papers [**3, 4**] are different from those of the present paper, but both approaches rest upon certain numerical techniques in the classical theory of algebraic functions.

Schinzel [**14**] used Siegel's Theorem [**16**] to sharpen Runge's Theorem. His results do not lead to computable bounds, since they depend on Siegel's Diophantine approximation methods [**15**], and we, therefore, do not consider them here.

For more on Runge's Theorem see also Ellison [**1**], Maillet [**8, 9**], Mordell [**10**] and Skolem [**18**]. Mordell proves, but does not compute bounds for, the special case of an irreducible equation where the leading form F_1 is not a constant times a power of an irreducible polynomial. Skolem gives an informative historical discussion of the events up to 1938.

1.3. *Some algebraic concepts and notation.* An *algebraic number* θ is the solution of an equation $P(x) = 0$ where

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \qquad a_n > 0,$$

is an irreducible polynomial in $\mathbf{Z}[x]$. The polynomial P is the *defining polynomial* of θ and

$$\deg \theta = \deg P = n, \qquad \text{height } \theta = \text{height } P = \max_{0 \leq i \leq n} |a_i| .$$

An algebraic number is an *algebraic integer* if its defining polynomial is monic, that is $a_n = 1$. The *denominatior* of θ is the smallest positive integer, a, so that $a\theta$ is an algebraic integer. The denominator is a divisor of a_n.

The *conjugates* of θ are the zeros $\theta^{(1)} = \theta, \theta^{(2)}, \ldots, \theta^{(n)}$ of P. The *norm* of θ is

$$\operatorname{norm} \theta = \theta^{(1)}\theta^{(2)} \cdots \theta^{(n)} = (-1)^n a_0/a_n.$$

An *algebraic number field* is a field $K = \mathbf{Q}(\theta)$ obtained by adjoining an algebraic number θ to $\mathbf{Q}$. The *degree* of K is

$$[K : Q] = \deg \theta.$$

The elements $\alpha \in K$ can be expressed uniquely as polynomials $f(\theta)$ where $f(x) \in \mathbf{Q}[x]$, $\deg f \leq [K : Q]$. The *K-conjugates* of α are $f(\theta^{(1)}), f(\theta^{(2)}), \ldots, f(\theta^{(n)})$ which are not necessarily distinct and the *K-norm* of α is

$$\operatorname{norm}_{K/Q} \alpha = f(\theta^{(1)}) f(\theta^{(2)}) \cdots f(\theta^{(n)}).$$

Treating the conjugates of θ as complex numbers we define the *house* of θ,

$$\overline{|\theta|} = \max_{1 \leq i \leq n} |\theta^{(i)}| .$$

Finally a few simple observations.

(i) Irreducibility in $\mathbf{Z}[x, y]$ means not only irreducibility of the polynomial in $\mathbf{Q}[x, y]$ but also that the coefficients are relatively prime.

(ii) For two polynomials $f(x)$, $g(x)$ we have

$$\text{height}(fg) \leq (1 + \deg f)\text{height}(f) \cdot \text{height}(g).$$

(iii) If $P(x) \in \mathbf{C}[x]$ and $P(\theta) = 0$ then

$$|\theta| < \text{height } P + 1.$$

(iv) A polynomial $P(x_1, x_2, \ldots, x_n) \in \mathbf{Z}[x_1, x_2, \ldots, x_n]$ which is symmetric in $x_1, x_2, \ldots, x_n$ is equal to a polynomial $Q(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbf{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n]$ in the elementary symmetric functions

$$\sigma_1 = x_1 + x_2 + \cdots + x_n, \sigma_2 = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n, \ldots, \sigma_n = x_1 x_2 \cdots x_n.$$

(v) Hadamard's inequality. If $A = (a_{ij})_{i, j = 1, 2, \ldots, n}$ is an $n \times n$ complex matrix then

$$|\det A| \leq \left(\prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

(vi) The notation $f(x) = O(g(x))$ means that $|f(x)| \leqslant c|g(x)|$ for some constant c and all large $|x|$. The symbol

$$\sum a_i x^i \ll \sum b_i x^i$$

means $|a_i| \leqslant b_i$ for all i.

For more on such algebraic concepts, see LeVeque [**7**, Volume II] and Pollard and Diamond [**11**].

2. Qualitative results.

2.1. THEOREM. *Let $y = f(x)$ where f is a function of a complex variable x given by a Puiseux series*

$$f(x) = \sum_{n=-m}^{\infty} a_n x^{-n/e}, \qquad m, e \in \mathbf{Z}, a_{-m} \neq 0, e > 0 \tag{2.2}$$

which converges for $|x| > R$. let $a_n \in K$, where K is an algebraic number field of degree s, for all $n \leqslant N$ and let M be the number of pairs of integers (μ, ν) with

$$1 \leqslant \mu \leqslant \nu m/e, \qquad 1 \leqslant \nu < se. \tag{2.3}$$

If

$$N \geqslant Me + (se - 1)m,$$

then the lattice points (x, y) on the curve $y = f(x)$ satisfy an equation $P(x, y) = 0$ where $P(x, y) \in \mathbf{Z}[x, y]$, $\deg_y P \leqslant se$, and P_λ is a monomial for all $\lambda \neq m/e$. For $\lambda = m/e$, P_λ is a constant multiple of a power of x times a power of an irreducible polynomial in $\mathbf{Z}[x, y]$. Hence, $\tilde{P} = P_{m/e}$. If the number of lattice points on $y = f(x)$ is infinite, then $P(x, f(x)) \equiv 0$.

PROOF. To exclude trivialities we observe that if $m < 0$ then $f(x) \to 0$ as $x \to \infty$. Thus for large x the only lattice points on $y = f(x)$ are given by the integral zeros of $f(x)$, and since $f(x)$ is not identically zero these are finite in number. If $m = 0$ then $f(x) \to a_0$ as $x \to \infty$ and either $f(x) \equiv a_0 \in \mathbf{Z}$ or, again, there are are only finitely many lattice points on $y = f(x)$. We, therefore, restrict our attention to the case $m > 0$ and hence $N \geqslant 0$.

Let $a_n^{(\sigma)}$; $\sigma = 1, 2, \ldots, s$; denote the conjugates of $a_n = a_n^{(1)}$, $n \leqslant N$ and let $\zeta = \exp(2\pi i/e)$. Consider the functions

$$F(x, y; \rho) = x^\rho \prod_{\sigma=1}^{s} \prod_{\varepsilon=0}^{e-1} \left(y - \sum_{n=-m}^{N} a_n^{(\sigma)} (\zeta^\varepsilon x^{-1/e})^n \right), \qquad \rho = 0, 1, \ldots, M. \tag{2.4}$$

For $y = f(x)$ we have

$$y - \sum_{n=-m}^{N} a_n^{(\sigma)} (\zeta^\varepsilon x^{-1/e})^n = \begin{cases} O(x^{-(N+1)/e}) & \text{for } \sigma = 1, \varepsilon = 0, \\ O(x^{m/e}) & \text{otherwise.} \end{cases} \tag{2.5}$$

Thus $F(x, y; \rho) = O(x^k)$ where

$$k \leqslant M + (se - 1)m/e - (N + 1)/e \leqslant -1/e$$

so that

$$F(x, y; \rho) = O(x^{-1/e}), \qquad \rho = 0, 1, \ldots, M. \tag{2.6}$$

Now, from (2.4) we see that

$$F(x, y; \rho) = P(x, y; \rho) + \sum_{\mu} \sum_{\nu} b_{\rho\mu\nu} y^{\nu}/x^{\mu} + O(x^{-1/e}) \tag{2.7}$$

where $P(x, y; \rho) \in \mathbf{Q}[x, y]$ and the double sum is extended over all μ, ν which satisfy (2.3). The $b_{\rho\mu\nu}$ are rational numbers. The terms collected in $O(x^{-1/e})$ are terms of the form y^{ν}/x^{μ} with $m\nu/e < \mu \leqslant Ns$, $0 \leqslant \nu \leqslant se$.

Since there are $M + 1$ functions $F(x, y; \rho)$, there exist integers B_{ρ}, not all zero, so that

$$\sum_{\rho=0}^{M} B_{\rho} b_{\rho\mu\nu} = 0 \tag{2.8}$$

for all pairs μ, ν which satisfy (2.3), and so that

$$Q(x, y) = \sum_{\rho=0}^{M} B_{\rho} P(x, y; \rho) = \left(\sum_{\rho=0}^{M} B_{\rho} x^{\rho} \right) y^{se} + \cdots \in \mathbf{Z}[x, y]. \tag{2.9}$$

From (2.7), (2.8), and (2.9) we get

$$\sum_{\rho=0}^{M} B_{\rho} F(x, y; \rho) = Q(x, y) + O(x^{-1/e}). \tag{2.10}$$

The lattice points (x, y) on $y = f(x)$ satisfy (2.6), and so by (2.10) they satisfy

$$Q(x, y) = O(x^{-1/e}). \tag{2.11}$$

Since $Q(x, y)$ is an integer, it follows from (2.11) that $Q(x, y) = 0$ for all lattice points with large $|x|$. For the finitely many x-coordinates of the remaining lattice points there is a polynomial $Q_1(x) \in \mathbf{Z}[x]$ so that $Q_1(x) = 0$ for all these values. Hence $P(x, y) = Q_1(x)Q(x, y)$ vanishes at all lattice points of (2.2).

If there are infinitely many lattice points then the analytic function $P(x, f(x))$ has infinitely many zeros in any neighborhood of ∞, while the point at ∞ is an algebraic singularity. Hence $P(x, f(x)) \equiv 0$.

Write $m/e = m_1/e_1$ where $(m_1, e_1) = 1$, and let $g(x) = 0$ be the defining equation of $a_{-m}^{e_1}$ with degree s_1 and leading coefficient b. Then

$$G(x, y) = x^{m_1 s_1} g(y^{e_1}/x^{m_1})$$

is irreducible in $\mathbf{Z}[x, y]$ and

$$P_{m/e}(x, y; \rho) = \tilde{P}(x, y; \rho) = c x^{\rho} G(x, y)^{es/e_1 s_1},$$

where $c = b^{-es/e_1 s_1}$. It follows that if ρ_1 is the largest index for which $B_{\rho_1} \neq 0$, then

$$Q_{m/e}(x, y) = \tilde{Q}(x, y) = c B_{\rho_1} x^{\rho_1} G(x, y)^{es/e_1 s_1}$$

and

$$\tilde{P}(x, t) = c B_1 x^{\rho_1} \tilde{Q}_1(x) G(x, y)^{es/e_1 s_1}.$$

This completes the proof of the theorem.

2.12. THEOREM (RUNGE). *Let $F(x, y)$ satisfy Runge's Condition. Then the Diophantine equation $F(x, y) = 0$ has only finitely many solutions $(x, y) \in \mathbf{Z}^2$.*

PROOF. If $\deg F = 1$, then the theorem is true because Runge's Condition is violated. We, therefore, assume that $\deg F > 1$. If $F(x, y) = 0$ has infinitely many integral solutions then there are infinitely many lattice points on one of the expansions $y = f(x)$ of y into a Puiseux series at ∞. By Theorem 2.1 this implies that $P(x, f(x)) = 0$ where P is the polynomial determined in Theorem 2.1. Since F is irreducible it must divide P and thus F_λ divides P_λ for all λ.

Hence F_λ is a monomial except possibly for a single value $\lambda = \lambda_0$, and F_{λ_0} is a constant times a power of x times a power of the irreducible $G(x, y)$ of Theorem 2.1.

Reversing the role of x and y we see that F_{λ_0} is a constant multiple of a power of y times a power of $G(x, y)$. Thus F_{λ_0} is either a monomial or a constant multiple of a power of $G(x, y)$.

If F_λ is a monomial for all λ then $\tilde{F}$ is a monomial. If $|x|$, $|y|$ are both large then a monomial $\tilde{F}$ dominates the rest of F. Thus $F = 0$ can have infinitely many solutions in this case only when $F = x - a$ or $F = y - b$, that is, when $\deg F = 1$. Therefore $\tilde{F} = F_{\lambda_0} = aG(x, y)^b$ where a, b are positive integers.

Note that Runge's Condition is not an invariant under changes of variables which preserve lattice points. Thus, say,

$$F(x, y) = (x + y^2)^2 - y^2 + ay + b$$

has leading part $(x + y^2)^2$ which is a power of an irreducible polynomial. But, if we set $x' = x + y^2$, $y' = y$ then $F(x, y) = G(x', y') = x'^2 - y'^2 + ay' + b$ where $\tilde{G} = (x' + y')(x' - y')$, so that G satisfies Runge's Condition. It would be interesting to characterize the class of polynomials which are equivalent to polynomials satisfying Runge's Condition under lattice-point preserving changes of variable.

All results in this section remain valid if we replace lattice points by lattice points over imaginary quadratic number fields. For lattice points over number fields with more than one Archimedean valuation we need additional hypotheses.

We use the following notation: Let L denote an algebraic number field of degree t, and let $\mathcal{O}_L$ denote the ring of integers of L. For $x \in L$, we have the conjugates $x^{(\tau)}$, $\tau = 1, 2, \ldots, t$. A pair $(x, y) \in \mathcal{O}_L^2$ is an *L-lattice point*.

2.13. THEOREM. *Let $f(x)$ be as in Theorem 2.1. Consider the L-lattice points (x, y) with the following properties*:

(i) $y^{(\tau_0)} = f(x^{(\tau_0)})$ *for some* τ_0 *with* $|x^{(\tau_0)}| \geqslant \lceil x \rceil^{c_1}$,

(ii) $\lceil y \rceil \leqslant c_2 \lceil x \rceil^{c_3}$,

(iii) $\lceil 1/x \rceil \leqslant c_4 \lceil x \rceil^{c_5}$,

where c_1, c_2, c_3, c_4 and c_5 are positive constants with

$$c_5 < c_1/es(t - 1).$$

Let M be as in Theorem 2.1. *Let*

$$N \geqslant (e + (se - 1)m)/(c_1 - c_5 se(t - 1)), \quad N \geqslant c_3 e/c_5, \quad N \geqslant m/c_5.$$

Then all the above L-lattice points satisfy an equation $P(x, y) = 0$, where P is a polynomial with the properties given in Theorem 2.1. *If the number of L-lattice points satisfying* (i)–(iii) *is infinite, then* $P(x, f(x)) \equiv 0$.

Note that Theorem 2.13 contains Theorem 2.1 as the special case $t = 1$, $c_1 = 1$, $c_3 = m/e$, $c_5 = m/N$.

PROOF. Construct the functions $F(x, y; \rho)$ of (2.4). For $|x^{(\tau_0)}| > R$. Condition (2.5) becomes

$$y^{(\tau_0)} - \sum_{n=-m}^{N} a_n^{(\sigma)}\left(\zeta^{\varepsilon}\left(x^{(\tau_0)}\right)^{-1/e}\right)^n = \begin{cases} O\left(\lceil x \rceil^{-c_1(N+1)/e}\right) & \text{for } \sigma = 1, \varepsilon = 0, \\ O\left(\lceil x \rceil^{m/e}\right) & \text{otherwise.} \end{cases}$$

Thus

$$F\left(x^{(\tau_0)}, y^{(\tau_0)}; \rho\right) = O\left(\lceil x \rceil^k\right), \qquad \rho = 0, 1, \ldots, M,$$

where

$$k \leqslant M + (se - 1)m/e - c_1(N + 1)/e.$$

For arbitrary τ we have from (ii) and (iii),

$$\begin{aligned} y^{(\tau)} - \sum_{n=-m}^{N} a_n^{(\sigma)}\left(\zeta^{\varepsilon}\left(x^{(\tau)}\right)^{-1/e}\right)^n &= O\left(\lceil x \rceil^{c_3} + \lceil x \rceil^{m/e} + \lceil x \rceil^{c_5 N/e}\right) \\ &= O\left(\lceil x \rceil^{c_5 N/e}\right), \end{aligned}$$

and hence

$$F\left(x^{(\tau)}, y^{(\tau)}; \rho\right) = O\left(\lceil x \rceil^{c_5 s N}\right).$$

Now construct $Q(x, y)$ as in (2.9) to get

$$Q\left(x^{(\tau_0)}, y^{(\tau_0)}\right) = O\left(\lceil x \rceil^k\right), \qquad Q\left(x^{(\tau)}, y^{(\tau)}\right) = O\left(\lceil x \rceil^{c_5 s N}\right).$$

Thus

$$\operatorname{norm} Q(x, y) = O\left(\lceil x \rceil^{k + c_5 s N(t-1)}\right) = O\left(\lceil x \rceil^{-c_1/e}\right),$$

and consequently $Q(x, y) = 0$ for all but a finite number of our L-lattice points. The construction of $P(x, y)$ now proceeds as in the proof of Theorem 2.1. If there are infinitely many such L-lattice points then $|x|$, and hence $x^{(\tau_0)}$, tend to infinity so that $P(x, f(x))$ has infinitely many zeros in every neighborhood of ∞. Thus $P(x, f(x)) \equiv 0$.

We can now apply Theorem 2.13 to the following generalization of Runge's Theorem.

2.14. THEOREM. *Let $F(x, y)$ be irreducible in $\mathbf{Z}[x, y]$ and set*

$$F(x, y) = \sum_{i=0}^{d_1} x^i f_i(y) = \sum_{j=0}^{d_2} y^j g_j(x)$$

where

$$g_{d_2}(x) = a \prod_{r=1}^{v} (x - \alpha_r)^{\mu_r}, \qquad f_{d_1}(y) = b \prod_{s=1}^{w} (y - \beta_s)^{\nu_s}$$

and assume that there exist infinitely many L-lattice points $(x, y) \in \mathcal{O}_L^2$ which satisfy $F(x, y) = 0$ and

(iv) $$\lceil 1/x \rceil \leqslant c_6 \lceil x \rceil^{1/(t-1)d_2^2}, \qquad \lceil 1/y \rceil \leqslant c_6 \lceil y \rceil^{1/(t-1)d_1^2}$$

where c_6 is a positive constant. Assume one of the following conditions for these L-lattice points:

(v)
$$\overline{|1/(x-\alpha_r)|} \leqslant c_7 \overline{|y|}^{1/\mu_r - c_8}, \qquad r = 1, 2, \ldots, v,$$
$$\overline{|1/(y-\beta_s)|} \leqslant c_7 \overline{|x|}^{1/\nu_s - c_8}, \qquad s = 1, 2, \ldots, w,$$

where c_7, c_8 are positive constants. Then $F(x, y)$ violates Runge's Condition.

PROOF. Assume that there are infinitely many L-lattice points (x, y) which satisfy the hypotheses of the theorem. Then, since there are only finitely many integers θ in $\mathcal{O}_L$ with $\overline{|\theta|}$ below a fixed bound, we have $\overline{|x|} \to \infty$ and $\overline{|y|} \to \infty$. Since $F(x^{(\tau)}, y^{(\tau)}) = 0$ for $\tau = 1, 2, \ldots, t$ we can pick τ_0 so that $|x^{(\tau_0)}| = \overline{|x|}$ for infinitely many values of x. We can also pick one Puiseux series expansion $y = f(x)$ at ∞ with $y^{(\tau_0)} = f(x^{(\tau_0)})$ for infinitely many of those values of x. If we restrict attention to that series and to those L-lattice points then (i) of Theorem 2.13 is satisfied with $c_1 = 1$. (iii) of Theorem 2.13 follows from (iv) since the degree s of the field of Puiseux series coefficients of f satisfies $s \leqslant d_2$, and since $e \leqslant d_2$, where this last inequality is strict if Runge's Condition holds. To verify (ii) we note that (v) implies

$$\overline{|y|}^{d_2 + c_9 - 1} = O\left(\overline{|y|}^{d_2 - 1} \cdot \overline{|x|}^{d_1}\right)$$

and hence

$$\overline{|y|} = O\left(\overline{|x|}^{d_1/c_8}\right)$$

which implies (ii). Thus, according to Theorem 2.13 we have an equation $P(x, f(x)) \equiv 0$ where $\tilde{P}(x, y)$ is a constant times a power of x times a power of an irreducible polynomial.

We can now interchange the role of x and y to get that $\tilde{F}$ divides both a constant times a power of x times a power of an irreducible polynomial and a constant times a power of y times a power of an irreducible polynomial. Thus, in order to show that F violates Runge's Condition it suffices to show that $\tilde{F}$ is not a monomial.

Now, if $\tilde{F} = a_{d_1 d_2} x^{d_1} y^{d_2}$ then

$$|x^{(\tau)}|^{d_1} |y^{(\tau)}|^{d_2} = O\left(|x^{(\tau)}|^{d_1} |y^{(\tau)}|^{d_2 - 1} + |x^{(\tau)}|^{d_1 - 1} |y^{(\tau)}|^{d_2}\right).$$

Thus not both $|x^{(\tau)}|$ and $|y^{(\tau)}|$ can be large. Since we are interested in solutions with $|x^{(\tau_0)}| = \overline{|x|} \to \infty$ we must have $y^{(\tau_0)}$ bounded. Now (v) implies

$$\overline{|x|}^{d_1 + c_8 - 1} = O\left(\overline{|x|}^{d_1 - 1}\right)$$

which is incompatible with the unboundedness of $\overline{|x|}$.

(v) in Theorem 2.14 is necessary. For example, the equation $(x - a)(y - b) = 1$ violates Runge's Condition, but has infinitely many L-lattice point solutions $x = a + \eta$, $y = b + \eta^{-1}$, in any field L with an infinite group of units η.

3. Quantitative results. As before we consider Diophantine equations

(3.1) $$F(x, y) = 0, \qquad (x, y) \in \mathbf{Z}^2,$$

where F is an irreducible polynomial in $\mathbf{Z}[x, y]$ and

$$F = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} a_{ij} x^i y^j,$$

$d_1 = \deg_x F$, $d_2 = \deg_y F$, $d = \max\{d_1, d_2\}$, $h =$ height F.

Equation (3.1) defines y as an algebraic function of the complex variable x. The irreducibility of F implies that the resultant $\operatorname{res}_y(F, F_y)$ of F and $F_y = \partial F/\partial y$ is a polynomial in x which does not vanish identically. The finite singularities x of y satisfy $\operatorname{res}_y(F, F_y) = 0$.

In this section we compute bounds for certain equations (3.1) that satisfy Runge's Condition. We begin with some simple cases.

3.2. THEOREM. *If $\tilde{F} = a_{d_1 d_2} x^{d_1} y^{d_2}$ is a monomial, $d_1, d_2 > 0$, then the integral solutions of* (3.1) *satisfy*

$$\max\{|x|, |y|\} \leqslant (2(h+1))^{d+1}.$$

PROOF. For $|x| > 1$, $|y| > 1$, we have

$$\begin{aligned} |F(x,y)| &\geqslant |a_{d_1 d_2} x^{d_1} y^{d_2}| - \sum_{(i,j)\neq(d_1,d_2)} \sum |a_{ij} x^i y^j| \\ &\geqslant |x|^{d_1} |y|^{d_2} - h\left(\sum_{i=0}^{d_1} |x^i| \cdot \sum_{j=0}^{d_2} |y^j| - |x|^{d_1} |y|^{d_2} \right) \\ &= (h+1)|x|^{d_1}|y|^{d_2} - h\frac{|x|^{d_1+1} - 1}{|x| - 1}\frac{|y|^{d_2+1} - 1}{|y| - 1} \\ &> \frac{|x|^{d_1}|y|^{d_2}}{(|x|-1)(|y|-1)}\left(|x||y| - (h+1)(|x| + |y|)\right). \end{aligned}$$

Thus $|F(x,y)| > 0$ if $|x|, |y| \geqslant 2(h+1)$. If we assume that $|x_0| < 2(h+1)$ then the equation $F(x_0, y) = 0$ is an equation for y which does not hold identically and has height less than $(2(h+1))^{d+1}$. Hence all solutions satisfy $|y| \leqslant (2(h+1))^{d+1}$. Similarly, if $|y_0| < 2(h+1)$ then the solutions of $F(x, y_0) = 0$ satisfy $|x| \leqslant (2(h+1))^{d+1}$.

3.3. THEOREM. *If $\tilde{F}$ is reducible in $\mathbf{Z}[x, y]$, and if either $d_1 = 1$ or $d_2 = 1$, then the integral solutions of* (3.1) *satisfy*

$$\max\{|x|, |y|\} \leqslant d(h+1)^{2d}.$$

Note that Runge's Condition is equivalent to the reducibility of $\tilde{F}$ in this case.

PROOF. Assume $d_2 = 1$. Then y is a rational function of x which is not a polynomial. We set $y = P(x)/Q(x)$ where $P(x), Q(x) \in \mathbf{Z}[x]$ and where

$$Q(x) = b_0 + b_1 x + \cdots + b_q x^q, \qquad |b_i| \leqslant h, \ (0 \leqslant i \leqslant q), \ b_q > 0, \ q \leqslant d.$$

If we write $X = b_q x$ then

$$Q(x) = b_q^{-q+1}\left(B_0 + B_1 X + \cdots + B_{q-1} X^{q-1} + X^q\right)$$

where $B_i = b_i b_q^{q-1-i}$; $i = 0, 1, \ldots, q$. Hence

$$\begin{aligned} Q(x)^{-1} &\ll b_q^{q-1} X^{-q}\left[1 - h\left(b_q X^{-1} + b_q^2 X^{-2} + \cdots + b_q^q X^{-q}\right)\right]^{-1} \\ &\ll b_q^{q-1} X^{-q} / \left(1 - (h+1) b_q X^{-1}\right) = b_q^{-1} x^{-q} / \left(1 - (h+1)/x\right). \end{aligned}$$

Since the coefficients in the Laurent series of $Q(x)^{-1}$ in terms of X are integers divisible by b_q^{q-1}, it follows that the coefficient of $x^{-j}, j \geqslant q$, in the Laurent series of $Q(x)^{-1}$ in terms of x is a rational number with denominator b_q^{j-q+1}.

We now write

$$P(x) = a_0 + a_1 x + \cdots + a_p x^p \ll h(1 + x + \cdots + x^d).$$

Thus the nonnegative powers of x in the Laurent series of P/Q form a polynomial $b_q^{-d+q-1}R(x)$ where $R(x) \in \mathbf{Z}[x]$, deg $R < d$ and

$$\text{height } R \leqslant b_q^{d-q}(h+1)^d.$$

The negative powers in the Laurent series P/Q form a fraction S/Q where

$$S = P - b_q^{-d+q-1}QR.$$

Thus $b_q^{d-q+1}S \in \mathbf{Z}[x]$ and

$$\text{height}\left(b_q^{d-q+1}S\right) \leqslant b_q^{d-q}\left(b_q h + dh(h+1)^d\right). \tag{3.4}$$

Now, if x, y are integers then $b_q^{d-q+1}S(x)/Q(x)$ is an integer. Thus either $S(x) = 0$, which according to (3.4) implies

$$|x| \leqslant b_q^{d-q}d(h+1)^{d+1} < d(h+1)^{2d},$$

or $|Q(x)| \leqslant b_q^{d-q+1}|S(x)|$, which yields

$$b_q|x|^q - h\frac{|x|^q - 1}{|x| - 1} \leqslant |Q(x)| \leqslant b_q^{d-q+1}|S(x)| \leqslant b_q^{d-q}d(h+1)^{d+1}\frac{|x|^q - 1}{|x| - 1}.$$

Hence

$$|x| \leqslant b_q^{-1}\left(b_q^{d-q}d(h+1)^{d+1} + h\right) + 1 < d(h+1)^{2d}.$$

Therefore attention can now be restricted to the case that $d_1, d_2 > 1$ and $\tilde{F}$ is not a monomial. This general case requires an analysis of the Puiseux series expansions of y. Such Puiseux series converge and represent y for all finite values of x in the exterior of any circle about the origin that encloses all finite singularities of y.

3.5. Lemma. *An algebraic function y defined by* (3.1) *has no singularities with $R_0 \leqslant |x| < \infty$, where*

$$R_0 = 4^d d^{4d} h^{2d-1}. \tag{3.6}$$

Proof. Let x be a finite singularity of y. Then

$$D(x) = \text{res}_y(F, F_y) = 0.$$

Here $D(x)$ is an $n \times n$ determinant, with $n \leqslant 2d - 1$, whose nonzero entries are the coefficients of the powers of y in F and F_y. Thus the entries are polynomials of degree at most d, and height at most h for the coefficients of F and at most dh for the coefficients of F_y. So, using Hadamard's inequality, we have

$$D(x) \ll (2d-1)^{d-1/2}d^d h^{2d-1}(1 + x + \cdots + x^d)^{2d-1}.$$

Therefore

$$\text{height } D \leqslant (2d-1)^{d-1/2}d^d h^{2d-1}(1+d)^{2d-1} < 4^d d^{4d} h^{2d-1}.$$

Now $D(x) = 0$ implies

$$|x| < \text{height } D + 1 \leq 4^d d^{4d} h^{2d-1}.$$

Let

$$y = \sum_{n=-m}^{\infty} a_n x^{-n/e} \tag{3.7}$$

be a Puiseux series for a function defined by (3.1), with coefficients a_n in a field K, $[K : \mathbf{Q}] = s$. Since all the conjugate series

$$y(x, \sigma, \varepsilon) = \sum_{n=-m}^{\infty} a_n^{(\sigma)} (\zeta^{\varepsilon} x^{-1/e})^n, \tag{3.8}$$

$\sigma = 1, 2, \ldots, s$; $\varepsilon = 0, 1, \ldots, e-1$; $\zeta = \exp(2\pi i/e)$, are also solutions of (3.1), it follows that the number of distinct series (3.8) is at most d_2. Moreover, it follows from Runge's Condition, by employing an argument similar to the proof of Theorem 2.12, that the number of distinct series (3.8) is, in fact, less than $d_2 \leq d$. In particular it follows, as pointed out earlier, that $e < d_2$. We assume that Runge's Condition holds for the Diophantine equation (3.1) in the computation of the bounds to follow.

We now normalize (3.7) by setting $x = t^{-e}$, $y = z/t^m$. Then the series

$$z = \sum_{n=0}^{\infty} b_n t^n, \qquad b_n = a_{n-m}, \tag{3.9}$$

satisfies the equation

$$H(t, z) = t^k F(t^{-e}, z/t^m) = 0,$$

where $k \leq d(e+m)$ is chosen so that H is a polynomial and $H(0, z) \neq 0$. Since $1 \leq e \leq d$ and $0 \leq m \leq d$ (the case $m < 0$ is trivial), it follows that

$$\text{height } H = h, \quad \deg_t H < 2d^2, \quad \deg_z H \leq d.$$

Moreover, z is analytic for $|t| \leq R_0^{-1/e}$ where R_0 is given by (3.6).

Each of the formal power series

$$z_\sigma = \sum_{n=0}^{\infty} b_n^{(\sigma)} t^n, \qquad \sigma = 1, 2, \ldots, s,$$

is analytic for $|t| \leq R_0^{-1/e}$. Thus

$$|b_n^{(\sigma)}| = \left| \frac{1}{2\pi i} \oint_{|t|=r} \frac{z_\sigma(t)}{t^{n+1}} dt \right| \leq \frac{1}{r^n} \max_{|t|=r} |z_\sigma(t)|, \qquad r \leq R_0^{-1/e}. \tag{3.10}$$

In order to estimate $z_\sigma(t)$ we write

$$H(t, z) = A_0(t) + A_1(t) z + \cdots + A_{d_2}(t) z^{d_2}. \tag{3.11}$$

Then $\deg A_i(t) < 2d^2$, height $A_i \leq h$ and, since $r \leq (4h)^{-1}$, we have

$$\max_{|t|=r} |A_i(t)| < h + 1, \qquad i = 0, 1, \ldots, d_2.$$

Also, writing $A_{d_2}(t) = t^l B(t)$ with $B(0) \neq 0$ we have

$$\min_{|t|=r} |A_{d_2}(t)| \geq r^l \min_{|t|=r} B(t) > \frac{1}{2} r^{2d^2}.$$

Thus, if we divide (3.11) by $A_{d_2}(t)$ we get a monic equation for $z_\sigma(t)$ of height at most $2(h+1)r^{-2d^2}-1$. Hence

$$\max_{|t|=r}|z_\sigma(t)|< 2(h+1)r^{-2d^2}$$

and (3.10) yields

$$\left|b_n^{(\sigma)}\right| \leqslant 2(h+1)r^{-2d^2-n}.$$

If we choose $r = R_0^{-1/e}$ we get

$$\boxed{b_n} \leqslant 2(h+1)R_0^{2d^2/e}R_0^{n/e} = C_1R_1^n \tag{3.12}$$

where

$$C_1 = 2(h+1)R_0^{2d^2/e}, \qquad R_1 = R_0^{1/e}.$$

Substitution of (3.12) in the Puiseux series (3.8) yields

$$y(x;\sigma,\varepsilon) \ll C_1R_1^m \sum_{n=-m}^{\infty} \left(R_1x^{-1/e}\right)^n = C_1x^{m/e}/\left(1-(R_0/x)^{1/e}\right). \tag{3.13}$$

In order to get a better quantitative estimate we now modify the definition (2.4) of $F(x, y; \rho)$. Let $\rho = 0, 1, \ldots, 2M$ and construct the functions $F(x, y; \rho)$ from an algebraic function y given by (3.1). We let $N = \infty$ since now all of the Puiseux coefficients a_n are in an algebraic number field. Let

$$F(x, y; \rho) = x^\rho \prod \left(y - \sum_{n=-m}^{\infty} a_n^{(\sigma)}(\zeta^\varepsilon x^{-1/e})^n \right), \tag{3.14}$$

where the product is extended over the less than d series (3.8) that are *distinct*.

By using (3.13) we get

$$\begin{aligned} F(x, y; \rho) &\ll x^\rho\left(y + C_1x^{m/e}\left(1-(R_0/x)^{1/e}\right)^{-1}\right)^{d-1} \\ &\ll \sum_{\delta=0}^{d-1} y^\delta \binom{d}{\delta} C_1^{d-\delta}x^{\rho+(d-\delta)m/e}\left(1-(R_0/x)^{1/e}\right)^{\delta-d} \\ &\ll (2C_1)^d \sum_{\delta=0}^{d} y^\delta x^{\rho+(d-\delta)m/e}\left(1-(R_0/x)^{1/e}\right)^{-d} \\ &\ll (2C_1)^d \sum_{\delta=0}^{d} y^\delta x^{\rho+(d-\delta)m/e} \sum_{n=0}^{\infty}\binom{d+n-1}{d-1}(R_0/x)^{n/e} \\ &\ll (4C_1)^d \frac{x^{\rho+dm/e}}{1-2(R_0/x)^{1/e}} \sum_{\delta=0}^{d} y^\delta x^{-\delta m/e}. \end{aligned} \tag{3.15}$$

3.16. LEMMA. *The coefficient $b_0 = a_{-m}$ in (3.9) is algebraic and satisfies* $\deg b_0 \leqslant d$, $\boxed{b_0} < h/D_0 + 1$, *D_0b_0 is an algebraic integer where D_0 is a positive integer, with $D_0 \leqslant h$.*

PROOF. We have the nontrivial equation $H(0, b_0) = 0$ whose degree is at most d and whose leading coefficient is $D_0 \leqslant h$. The roots $b_0^{(i)}$ then satisfy $|b_0^{(i)}| < h/D_0 + 1$.

In the general case the computation of the denominators of b_n involve difficulties which we defer to §4. In this section we shall be concerned with the following case.

3.17. ASSUMPTION. Let $\beta = H_z(0, b_0) \neq 0$.

3.18. LEMMA. *Let θ be an algebraic number of degree s and let the leading coefficient in the defining equation of θ be D. If $P(x_1, x_2, \ldots, x_s)$ is a symmetric polynomial, with integral coefficients, in $x_1, x_2, \ldots, x_s$ and* $\deg_{x_1} P = n$ *then* $D^n P(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(s)}) \in \mathbf{Z}$.

PROOF. We have $P(x_1, x_2, \ldots, x_s) = Q(\sigma_1, \sigma_2, \ldots, \sigma_s)$ where $Q(y_1, y_2, \ldots, y_s) \in \mathbf{Z}[y_1, y_2, \ldots, y_s]$ and $\sigma_1, \sigma_2, \ldots, \sigma_s$ are the elementary symmetric functions of $x_1, x_2, \ldots, x_s$. Since $\deg Q = n$ and $D\sigma_i(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(s)}) \in \mathbf{Z}$ it follows that $D^n Q(\sigma_1, \sigma_2, \ldots, \sigma_s) \in \mathbf{Z}$ when $x_i = \theta^{(i)}$; $i = 1, 2, \ldots, s$.

3.19. LEMMA. *The quantity β of Assumption* 3.17 *satisfies an algebraic equation over $\mathbf{Z}$ whose leading coefficient is D_0^{d-1} and*

$$B = |\operatorname{norm} \beta| \leq (d+1)^d (2h)^{d^2} / D_0^{d(d-1)}.$$

PROOF. Write

$$\beta = H_z(0, b_0) = c_0 + c_1 b_0 + \cdots + c_{d-1} b_0^{d-1}$$

where $c_j \in \mathbf{Z}$, $|c_j| \leq (j+1)h$. Thus

$$\lceil\beta\rceil \leq h\left(1 + 2\lceil b_0\rceil + \cdots + d\lceil b_0\rceil^{d-1}\right). \tag{3.20}$$

Now for $x > 1$ we have

$$1 + 2x + \cdots + dx^{d-1} = \frac{d}{dx}\frac{x^{d+1}-1}{x-1} < (d+1)\frac{x^d}{x-1}.$$

Hence it follows from Lemma 3.16 and (3.20) that

$$\begin{aligned} \lceil\beta\rceil &< h(d+1)((h/D_0)+1)^d/(h/D_0) = (d+1)D_0(h/D_0+1)^d \\ &= (d+1)D_0^{-d+1}(h+D_0)^d < (d+1)(2h)^d/D_0^{d-1}, \end{aligned}$$

and consequently that

$$B \leq |\beta|^d < (d+1)^d (2h)^{d^2} / D_0^{d(d-1)}.$$

3.21. LEMMA. *If $\beta \neq 0$ then $b_n \in \mathbf{Q}(b_0)$ for all $n = 1, 2, \ldots,$ and $\beta^{2n-1} b_n$ is a polynomial, with integral coefficients, in b_0, of degree at most* $(2n-1)d$.

PROOF. Substitute $z = \sum_{n=0}^{\infty} b_n t^n$ in $H(t, z)$ and expand in powers of t:

$$H(t, z) = H_0 + H_1 t + \cdots + H_n t^n + \cdots.$$

Here H_n is a polynomial, with integral coefficients, in $b_0, b_1, \ldots, b_n$, where for each monomial $b_0^{k_0} b_1^{k_1} \cdots b_n^{k_n}$ we have

$$k_0 + k_1 + \cdots + k_n \leq d, \qquad k_1 + 2k_2 + \cdots + nk_n \leq n. \tag{3.22}$$

The equations $H_n = 0$ successively define b_n since

$$H_n = \beta b_n + I_n(b_0, b_1, \ldots, b_{n-1})$$

where I_n is a polynomial with integral coefficients.

We now proceed by induction on n. For $n = 1$ we have

$$\beta b_1 + H_t(0, b_0) = 0.$$

Thus $\beta b_1 = -H_t(0, b_0)$ is a polynomial, with integral coefficients, of degree at most d in b_0.

Now assume the lemma true for indices less than n. The terms in $I_n(b_0, b_1, \dots, b_{n-1})$ are integral multiples of monomials $b_0^{k_0} b_1^{k_1}, \dots, b_{n-1}^{k_{n-1}}$ where, by (3.22),

$$1k_1 + 3k_2 + \cdots + (2n-3)k_{n-1} = 2(k_1 + 2k_2 + \cdots + (n-1)k_{n-1}) - (k_1 + k_2 + \cdots + k_{n-1}) \leqslant 2n - 2,$$

since $k_1 + 2k_2 + \cdots + (n-1)k_{n-1} = n$ implies $k_1 + k_2 + \cdots + k_{n-1} \geqslant 2$ for all $n > 1$. Thus

$$\beta^{2n-1} b_n = -\beta^{2n-2} I_n(b_0, b_1, \dots, b_{n-1})$$

is a sum of integral multiples of monomials

$$\beta^l b_0^{k_0} (\beta b_1)^{k_1} (\beta^3 b_2)^{k_2} \cdots (\beta^{2n-3} b_{n-1})^{k_{n-1}},$$
$$l = 2n - 2 - (1k_1 + 3k_2 + \cdots + (2n-3)k_{n-1}) \geqslant 0,$$

which are, by the induction hypothesis, polynomials, with integral coefficients, in b_0 of degree at most

$$(2n - 2 - 1k_1 - 3k_2 - \cdots - (2n-3)k_{n-1})d + k_0 + (1k_1 + 3k_2 + \cdots + (2n-3)k_{n-1})d = (2n-2)d + k_0 \leqslant (2n-1)d.$$

We now employ a modified version of the argument in the proof of Theorem 2.1. Define the functions $F(x, y; \rho)$, for $\rho = 0, 1, \dots, 2M$, as in (3.14). It follows from Assumption 3.17 that $a_n \in \mathbf{Q}(b_0)$, for all n. Now $\deg_y F(x, y; \rho) < d$ and the quantity M in (2.3) becomes the number of lattice points (μ, ν) with

$$1 \leqslant \mu \leqslant \nu m/e, \quad 1 \leqslant \nu < d.$$

So

$$M \leqslant md(d-1)/2e.$$

The estimation (2.6) is replaced by

$$F(x, y; \rho) = 0, \qquad \rho = 0, 1, \dots, 2M.$$

Now, as in (2.7), write

$$F(x, y; \rho) = P(x, y, \rho) + \sum_\mu \sum_\nu b_{\rho\mu\nu} y^\nu / x^\mu + O(x^{-1/e}).$$

Then the coefficients $b_{\rho\mu\nu}$ are symmetric polynomials, with integral coefficients, in the conjugates of b_n where

$$m + 1 \leqslant n \leqslant (d-1)m + \rho e + m \leqslant dm + 2eM \leqslant md^2.$$

The degree in b_n is at most e. Thus, according to Lemma 3.21 the quantities

$$B^{(2md^2-1)e} b_{\rho\mu\nu} = B_{\rho\mu\nu}$$

are symmetric polynomials, with integral coefficients, in the conjugates of b_0, of degree at most $(2md^2 - 1)de$ in b_0. Hence, by Lemma 3.18,

$$B^*_{\rho\mu\nu} = D_0^{(2md^2-1)de} B_{\rho\mu\nu} \in \mathbf{Z}.$$

Also, by Lemma 3.19,

$$|B^*_{\rho\mu\nu}| \leqslant D_0^{(2md^2-1)de} |B_{\rho\mu\nu}| \leqslant \left(D_0^d B\right)^{(2md^2-1)e} |b_{\rho\mu\nu}|$$
$$\leqslant \left[(d+1)(2h)^d / D_0^{d-2}\right]^{(2md^2-1)de} |b_{\rho\mu\nu}|.$$

From (3.15) we infer that

$$|b_{\rho\mu\nu}| < (4C_1)^d \left(2R_0^{1/e}\right)^{2Me+dm} \leqslant (4C_1)^d \left(2R_0^{1/e}\right)^{md^2}.$$

It follows that

$$|B^*_{\rho\mu\nu}| < d^{4d^5}(2h)^{3d^6}. \tag{3.23}$$

We now need Siegel's Lemma, whose proof is included for the sake of completeness.

3.24. LEMMA. *The system of equations*

$$\sum_{j=0}^{2M} A_{ij}x_j = 0, \quad |A_{ij}| \leqslant A, \quad A_{ij} \in \mathbf{Z}, \qquad i = 1, 2, \ldots, M,$$

has nontrivial integral solutions satisfying

$$|x_j| < (2M+1)A, \qquad j = 0, 1, \ldots, 2M.$$

PROOF. Set

$$y_i = \sum_{j=0}^{2M} A_{ij}x_j, \qquad i = 1, 2, \ldots, M,$$

and let x_j vary over the integers $0, 1, \ldots, X$. Then $-N_i X \leqslant y_i \leqslant P_i X$ where

$$-N_i = \sum_{\substack{j=0 \\ A_{ij}<0}}^{2M} A_{ij}, \quad P_i = \sum_{\substack{j=0 \\ A_{ij}>0}}^{2M} A_{ij}, \quad P_i + N_i \leqslant (2M+1)A.$$

Thus there are $(X+1)^{2M+1}$ choices for the vector $\mathbf{x} = (x_0, x_1, \ldots, x_{2M})$ and no more than

$$((2M+1)AX+1)^M < ((2M+1)A)^M (X+1)^M$$

choices for the $\mathbf{y}$-vectors. If

$$(X+1)^{2M+1} \geqslant ((2M+1)A)^M (X+1)^M,$$

which certainly holds if

$$X + 1 \geqslant (2M+1)A,$$

then two distinct $\mathbf{x}$-vectors, $\mathbf{x}_1$ and $\mathbf{x}_2$, give the same $\mathbf{y}$-vector and the vector $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ yields the desired solution.

Using (3.23) we get the following

3.25. COROLLARY. *There exist integers C_ρ; $\rho = 0, 1, \dots, 2M$, not all zero, so that*

$$\sum_{\rho=0}^{2M} C_\rho B^*_{\rho\mu\nu} = 0, \qquad 1 \leqslant \mu \leqslant \nu m/e,\ 1 \leqslant \nu < d,$$

and

$$|C_\rho| < md^2 \max |B^*_{\rho\mu\nu}| < d^{5d^5}(2h)^{3d^6}.$$

For the C_ρ in Corollary 3.25 we have

$$\sum_{\rho=0}^{2M} C_\rho F(x, y; \rho) = Q(x, y) + O(x^{-1/e}) \tag{3.26}$$

where

$$Q(x, y) = \sum_{\rho=0}^{2M} C_\rho P(x, y; \rho)$$

is a polynomial, whose coefficients are symmetric polynomials with integral coefficients in the conjugates of b_n, $0 \leqslant n \leqslant 2Me + m < md^2$.

Thus, as before, we have

$$Q^*(x, y) = \left(D_0^d B\right)^{(2md^2-1)e} Q(x, y) \in \mathbf{Z}[x, y]$$

and, by (3.15),

$$\begin{aligned}\text{height } Q^* &< \left((d+1)^d (2h)^{d^2}\right)^{(2md^2-1)e} (2M+1) d^{5d^5}(2h)^{2d^6} \\ &\quad \cdot (4C_1)^d \left(2R_0^{1/e}\right)^{2Me+md} < d^{8d^5}(3h)^{5d^6}.\end{aligned}$$

To sum up we have the following

3.27. LEMMA. *The polynomial Q^* satisfies*

$$\deg_y Q^* < d, \quad \deg_x Q^* \leqslant 2M + md/e \leqslant d^3/e, \quad \text{height } Q^* < d^{8d^5}(3h)^{5d^6}.$$

The error term in (3.26) consists of those terms y^μ/x^ν in the sum on the left for which $\mu m - \nu e < 0$. From (3.13) we infer, for $|x| \geqslant 2^d R_0 > 2^e R_0$, that

$$|y| < 2C_1 |x|^{m/e}. \tag{3.28}$$

Hence (3.15) implies, for $|x| \geqslant 2^d R_0$, that the error terms, $E(x, y; \rho)$ ($\rho = 0, 1, \dots, 2M$), which are $O(x^{-1/e})$, in $F(x, y; \rho)$, satisfy

$$\begin{aligned}|E(x, y; \rho)| &< 4C_1^d \sum_{n=1}^{\infty} \left(2R_0^{1/e}\right)^{2Me+dm+n} |x|^{-n/e} (d+1)(2C_1)^d \\ &< 4d\left(8C_1^2\right)^d \left(2R_0^{1/e}\right)^{md^2+1} |x|^{-1/e}.\end{aligned}$$

Thus the error term, $E^*(x, y)$, in (3.26) satisfies

$$\begin{aligned}E^*(x, y) &\leqslant \sum_{\rho=0}^{2M} |C_\rho| |E(x, y; \rho)| \\ &< (2M+1) d^{5d^5}(2h)^{3d^6} \cdot 4d\left(8C_1^2\right)^d \left(2R_0^{1/e}\right)^{md^2+1} |x|^{-1/e} < 1,\end{aligned}$$

provided

$$|x| \geq (2M+1)^e d^{5d^5 e}(2h)^{3d^6 e}(4d)^e (8C_1^2)^{de}(2^e R_0)^{md^2+1}.$$

This certainly holds when

$$|x| \geq d^{5d^6}(2h)^{3d^7}. \tag{3.29}$$

Thus $Q^*(x, y) = 0$ for all lattice points on (3.7) which satisfy (3.29). All x-coordinates of these lattice points therefore satisfy

$$R(x) = \operatorname{res}_y(F(x, y), Q^*(x, y)) = 0.$$

The resultant $R(x)$ is a determinant with no more than $d - 1$ rows whose entries are coeffcients of powers of y in F, and no more than d rows whose entries are coefficients of powers of y in Q^*. So, by using Hadamard's inequality and Lemma 3.27, we get, $h(Q^*) = $ height Q^*,

$$\begin{aligned} R(x) &\ll (2d-1)^{d-1/2} h^{d-1} h(Q^*)^d (1 + x + \cdots + x^d)^{d-1}(1 + x + \cdots + x^{d^3})^d \\ &\ll (2d)^d h^{d-1} d^{8d^6}(3h)^{5d^7}(1 + x + \cdots + x^d)^{d-1}(1 + x + \cdots + x^{d^3})^d. \end{aligned}$$

Hence

$$\text{height } R < (2d)^d h^{d-1} d^{8d^6}(3h)^{5d^7}(d+1)^{d-1}(d^3+1)^d < d^{9d^6}(3h)^{6d^7}.$$

Therefore any zero of $R(x)$ satsisfies

$$|x| < d^{9d^6}(3h)^{6d^7}. \tag{3.30}$$

The same bound works for $|y|$ from (3.28), since for $e < m \leq d$ the bound (3.30) could be reduced substantially. Summing up we have the following theorem.

3.31. THEOREM. *Under Runge's Condition and Assumption* 3.17, *the solutions of the Diophantine equation* (3.1) *satisfy*

$$\max\{|x|, |y|\} \leq d^{9d^6}(3h)^{6d^7}.$$

4. The case $H_z(0, b_0) = 0$. Here the successive equations for the Taylor coefficients b_n, which result from the equation $H(t, z) = 0$, are not necessarily linear for all n; but they do become linear for all $n > n_0$. We develop the idea for general algebraic functions, following a method employed by Heine [**2**] (see Pólya and Szegö [**12**, Volume II, Problem VIII, §3]).

4.1. LEMMA. *Let $H(t, z)$ be an irreducible polynomial in* $\mathbf{C}[t, z]$ *with* $\deg_t H \leq d_0$ *and* $\deg_z H = d_2$. *Let*

$$D(t) = \operatorname{res}_z(H, H_z)$$

be the resultant of H and H_z. Then

$$\deg D(t) \leq d_0(2d_2 - 1).$$

Hence, if n_1 is the order of the zero of $D(t)$ at $t = 0$ then

$$n_1 \leq d_0(2d_2 - 1).$$

PROOF. The polynomial $D(t)$ is the value of a determinant of size no greater than $(2d_2 - 1) \times (2d_2 - 1)$, whose entries are polynomials in t of degree no greater than d_0.

4.2. LEMMA. *Let $z_1, z_2, \ldots, z_{d_2}$ be the solutions of $H(t, z) = 0$, analytic in a punctured neighborhood, $0 < |t| < \delta$, of 0. Let n_0 be the least integer so that*

$$\lim_{t \to 0} t^{-n_0 - 1}(z_i - z_j) = \infty \quad \text{for } 1 \leqslant i < j \leqslant d_2.$$

Then $n_0 \leqslant n_1/2 \leqslant d_0(d_2 - \frac{1}{2})$.

PROOF. Write $H(t, z) = A_0(t) + A_1(t)z + \cdots + A_{d_2}(t)z^{d_2}$. Then

$$D(t) = (-1)^{d_2(d_2-1)/2} A_{d_2}(t)^{2d_2 - 1} \prod_{1 \leqslant i < j \leqslant d_2} (z_i - z_j)^2.$$

If the lemma were false then one of the factors, $(z_i - z_j)^2$, would have a zero of order at least $n_1 + 1$ at $t = 0$. While some of the conjugates z_k may have poles at $t = 0$, these poles are cancelled by the zeros of $A_{d_2}(t)$. Thus it would follow that $D(t)$ has a zero of order greater than n_1 at $t = 0$, in contradiction to Lemma 4.1.

4.3. LEMMA. *Let $H(t, z)$ be as in Lemma 4.1 and n_0 as in Lemma 4.2. Let $z = \sum_{n=0}^{\infty} b_n t^n$ be a formal power series solution of $H(t, z) = 0$ which is therefore analytic in a neighborhood of 0. Then the successive equations for b_n in terms of $b_0, b_1, \ldots, b_{n-1}$ are linear of the form $Bb_n + C_n = 0$, provided $n > n_0$. Here B is a polynomial in $b_0, b_1, \ldots, b_{n_0}$ and C_n is a polynomial in $b_0, b_1, \ldots, b_{n-1}$.*

PROOF. Set

$$z = \sum_{n=0}^{n_0} b_n t^n + t^{n_0 + 1} w$$

and write

$$H(t, z) = t^{M_1} K(t, w)$$

where M_1 is chosen so that $K(t, w)$ is a polynomial in t and w and $K(0, w) \neq 0$. Then by Lemma 4.2 the solutions $w = w_1, w_2, \ldots, w_{d_2}$ of $K(t, w) = 0$ have the property that only w_1 is bounded as $t \to 0$ while $w_2, w_3, \ldots, w_{d_2}$ tend to ∞. Hence, if we write

$$K(t, w) = K_0(t) + K_1(t)w + \cdots + K_{d_2}(t)w^{d_2}, \tag{4.4}$$

then, expressing K_δ as elementary symmetric functions of $w_1, w_2, \ldots, w_{d_2}$, we get $K_2(0) = K_3(0) = \cdots = K_{d_2}(0) = 0$ while $B = K_1(0) \neq 0$. Then coefficients of the K_δ are polynomials in $b_0, b_1, \ldots, b_{n_0}$. If we now set $w = \sum_{n=n_0+1}^{\infty} b_n t^n$ and collect coefficients of t^n in (4.4) then we get an equation $Bb_n + C_n = 0$, where the terms collected in C_n involve only $b_0, b_1, \ldots, b_{n-1}$.

We summarize the contents of Lemmas 4.1–4.3, and Lemma 3.21, whose proof is also valid in the present setting $H(t, z) \in \mathbf{C}[t, z]$, in the form of a quantitative version of a classical result.

4.5. THEOREM. *Let $H(t, z)$ be an irreducible polynomial in $\mathbf{C}[t, z]$ with* $\deg_t H(t, z) \leq d_0$ *and* $\deg_z H(t, z) = d_2$. *Let*

$$z = \sum_{n=0}^{\infty} b_n t^n$$

satisfy $H(t, z) = 0$ in a neighborhood of zero. Then there is a value of the index $n = n_0$ for which all the coefficients b_n lie in the field generated by $b_0, b_1, \ldots, b_{n_0}$ and the coefficients of H. For $n > n_0$, the equation for b_n, obtained by substituting the power series expansion of z into $H(t, z) = 0$, is linear, of the form $Bb_n + C_n = 0$, where $B \neq 0$ is a polynomial in $b_0, b_1, \ldots, b_{n_0}$ and C_n is a polynomial in $b_0, b_1, \ldots, b_{n-1}$. The coefficients of B and C_n are in the field of the coefficients of $H(t, z)$. A sufficient condition for $n_0 = 0$ is that $H_z(0, b_0) \neq 0$. In any case,

$$n_0 \leq d_0\left(d_2 - \tfrac{1}{2}\right).$$

This last inequality has computational applications in the theory of algebraic functions (Hilliker [**5**]).

We now return to the Diophantine equation under consideration. Here $H(t, z) \in \mathbf{Z}[t, z]$ and $d_0 \leq d(e + m) \leq d(2d - 1)$, $d_2 \leq d$. Thus the constants n_0, n_1 satisfy

$$n_1 \leq d(2d - 1)(e + m) \leq d(2d - 1)^2, \qquad n_0 \leq n_1/2.$$

In order to complete our estimations we need an algebraic integer D^* so that $D^* b_n$ is an algebraic integer for $n = 0, 1, \ldots, n_0$. Then $(D^*)^{d_K}$ has algebraically integral coefficients.

We have, according to (3.11),

$$\text{(4.6)} \quad K_1(t) = t^{-M_1}\left[A_1(t) + 2A_2(t) \sum_{n=0}^{n_0} b_n t^n + \cdots + d_2 A_{d_2}(t)\left(\sum_{n=0}^{n_0} b_n t^n\right)^{d_2 - 1}\right].$$

The sum of the absolute values of the coefficients in each A_i is no larger than $(d + 1)h$ and, by (3.12),

$$\sum_{n=0}^{n_0} \lceil b_n \rceil < C_1 \sum_{n=0}^{n_0} R_1^n < 2C_1 R_1^{n_0}.$$

Thus (4.6) implies

$$\begin{aligned}
\lceil K_1(0) \rceil &< (d + 1)h \sum_{\delta=1}^{d-1} \left(2C_1 R_1^{n_0}\right)^{\delta} \\
&< 2(d + 1)h\left(2C_1 R_1^{n_0}\right)^{d-1} \\
&= 2(d + 1)h(2C_1)^{d-1} R_0^{n_0(d-1)/e}.
\end{aligned}$$

Now

$$\begin{aligned}
n_0(d - 1)/e &\leq d\left(d - \tfrac{1}{2}\right)(d - 1)(1 + m/e) \leq d\left(d - \tfrac{1}{2}\right)(d - 1)(d + 1) \\
&< d^4 - d^2.
\end{aligned}$$

Hence (3.12) and (3.6) give

$$\lceil K_1(0) \rceil < 2(d + 1)h\left(4(h + 1)R_0^{2d^2/e}\right)^{d-1} R_0^{d^4 - d^2} < \left(4d^4 h^2\right)^{2d^5}$$

Now, in a manner precisely analogous to the way in which we proved Lemma 3.21, we prove the following

4.7. LEMMA. *For each $n > 0$, the number $B_1^{2n-1} b_{n_0+n}$ is an algebraic integer, where*

$$B_1 = (D^*)^d K_1(0).$$

The estimation of the "denominators" is therefore complete once we estimate D^*. For this purpose we estimate algebraic integers D_n so that $D_n b_i$ is an algebraic integer for $i = 0, 1, \ldots, n$; $n \leqslant n_0$.

Assume that we have already computed D_n. Then write $z = (b_0 + b_1 t + \cdots + b_n t^n) + t^{n+1} z_n$ and construct the polynomial

$$L_n(t, z_n) = t^{-l_n} H(t, z)$$

where l_n is chosen so that L_n is a polynomial in t and z_n and $L_n(0, z_n) \neq 0$. The coefficients of $L_n(0, z_n)$ are polynomials, with integral coefficients, of degree at most d in $b_0, b_1, \ldots, b_n$ and hence

$$D_n^d L_n(0, b_{n+1}) = 0 \tag{4.8}$$

is an equation with integral coefficients for b_{n+1}. We, thus, can choose D_{n+1} as the leading coefficient in (4.8). As before we get

$$\overline{|D_{n+1}|} < (d+1) h \overline{|D_n|}^d \sum_{\delta=\mu}^{d} \binom{\delta}{\mu} (2 C_1 R_1^n)^{\delta-\mu}$$

where μ, $1 \leqslant \mu \leqslant d$, is the degree of (4.8) in b_{n+1}. Therefore

$$\overline{|D_{n+1}|} < \overline{|D_n|}^d (d+1) h \cdot 2^d \left(2 C_1 R_0^{n/e}\right)^{d-1} < C_2 \left(R_0^{n/e} \overline{|D_n|}\right)^d,$$

where

$$C_2 = (d+1) h 2^d (2C_1)^{d-1} < (4 d^2 h)^{4d^4/e}.$$

Hence

$$\overline{|D^*|} = \overline{|D_{n_0}|} < D_0^{d^{n_0-1}} C_2^{1+d+\cdots+d^{n_0-2}} R_1^{(n_0-1)d+(n_0-2)d^2+\cdots+d^{n_0-1}}.$$

Now

$$(n_0 - 1)d + (n_0 - 2)d^2 + \cdots + d^{n_0-1} = \frac{d^{n_0+1} - 1}{(d-1)^2} - \frac{n_0}{d-1} < 2 d^{n_0}$$

and therefore

$$\overline{|D^*|} < \left(C_2 R_0^{1/e} D_0\right)^{2d^{n_0}} < (4dh)^{d^{2d^3}}.$$

The rest of the argument is exactly as before. We omit details.

4.9. THEOREM. *If $F(x, y)$ satisfies Runge's Condition then the integral solutions x, y of $F(x, y) = 0$ satisfy*

$$\max\{|x|, |y|\} < (8dh)^{d^{2d^3}}.$$

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